Part I: Behrend's Function & DT Invariant

<u>Goal</u>:

- Define the constructible function ν_X.
- Sketch $\chi(X, \nu_X) = \int_{[X]^{\mathrm{vir}}} 1$

  For X admit a sym POT.

<u>Def</u> (The signed support of intrinsic normal cone)

For $X \hookrightarrow M$, M smooth, Let $C_{X/M}$ be normal cone , $\pi : C_{X/M} \longrightarrow X$

$$C_X := \sum_{c'} (-1)^{\dim \pi(c')} \mathrm{multi}(c') \cdot \pi(c')$$

c' runs over all irr component of $C_{X/M}$, and multiplicity taking w/r/t $c' \subset C_{X/M}$

Rmk : · The definition is intristic, and can be extend to X a DM Stack

· If X smooth, then $C_X = (-1)^{\dim X}[X]$

Def : $\nu_X := Eu(C_X)$, where Eu is the local Euler Obstruction.

$$Eu : Z_*(X) \xrightarrow{\sim} Con(X)$$

$\ni \sum_V m_V \mathbb{1}_V$

$m_V \in \mathbb{Z}$ finite sum.

Before define Eu, we need to introduce Nash blow-up:

- Take $V \hookrightarrow M'$ M' smooth

- $V^\circ \subset V$ smooth locus, Then we have map

$$\iota : V^\circ \longrightarrow \mathrm{Grass}_{M'}(\dim V, TM')$$

- Let $\tilde{V} := \overline{\mathrm{Im}(\iota)}$

- Now $\tilde{V}$ equipped with a vector bundle!

$T \to \tilde{V}$ $T|_{V^\circ}$ is the tangent bundle of V°

Back to Eu, we define it for any prime cycle $V \in Z_*(X)$, Let $\mu: \hat{V} \to V$ be the Nash blowup.

$$Eu(V)(p) := \int_{\mu^{-1}(p)} c(T) \cap s(\mu^{-1}(p), \hat{V})$$

Rmk: For V smooth, $Eu(V) = \mathbb{1}_V$

From the definition, $\nu_X = Eu(C_X)$ is constructible.

we list two properties:

i) If $X \to Y$ smooth, $f^* c_Y = (-1)^{\dim X/Y} c_X$

$f^* \nu_Y = (-1)^{\dim X/Y} \nu_X$

ii) $C_{X \times Y} = C_X \times C_Y$ so $\nu_{X \times Y} = \nu_X \boxdot \nu_Y$

$f \boxdot g\,(x, y) := f(x) \cdot g(y)$

Now we introduce the weighted Euler char:

<u>Def</u>: For $f = \sum m_V 1\!\!1_V \in Con(X)$,

$$\chi(X, f) = \sum m_V \chi(V) \qquad \chi([X/G]) = \frac{\chi(X)}{|G|}$$

and alternatively

$$\chi(X, f) = \sum n \cdot \chi(f^{-1}(n))$$

- For X scheme, $\chi(X, f)$ will be integer
- For X DM stack, $\chi(X, f)$ will be rational.

Like the usual topological Euler char, the weighted Euler char has the following properties:

i) $X = Z_1 \sqcup Z_2$ Z_1, Z_2 locally closed, then

$$\chi(X, f) = \chi(Z_1, f|_{Z_1}) + \chi(Z_2, f|_{Z_2})$$

ii) $\chi(X \times Y, f \boxtimes g) = \chi(X, f) \cdot \chi(Y, g)$

iii) If $X \to Y$ finite étale of deg d,

Then $\chi(X, f|_X) = d\,\chi(Y, f)$

For ν_X - weighted Euler char, we introduce the following notation :

<u>Def</u> : $\tilde{\chi}(X) := \chi(X, \nu_X)$

For $Z \hookrightarrow X$, $\tilde{\chi}(Z, X) := \chi(Z, \nu_X|_Z)$

<u>Rmk</u> : $\tilde{\chi}(X) \neq \tilde{\chi}(Z_1) + \tilde{\chi}(Z_2)$!

but we have the following properties :

i) : $Z = Z_1 \sqcup Z_2$ $\quad$ Z_1, Z_2 locally closed

$$\tilde{\chi}(Z, X) = \tilde{\chi}(Z_1, X) + \tilde{\chi}(Z_2, X)$$

ii) : $$\tilde{\chi}(Z_1 \times Z_2, X_1 \times X_2) = \tilde{\chi}(Z_1, X_1) \cdot \tilde{\chi}(Z_2, X_2)$$

iii) : For

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "\text{finite étale}"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "\text{smooth}"] \\
W \arrow[r] & Y
\end{tikzcd}$$

$$\tilde{\chi}(Z, X) = (-1)^{\dim X/Y} \deg(Z/W)\, \tilde{\chi}(W, Y)$$

Now we briefly review "local structure" of

sym POT :

<u>Prop</u> : The following is the Zariski / étale local

model for scheme / stack with sym POT :

$X \hookrightarrow M$, M smooth , $X = Z(\omega)$ for some

<u>almost closed</u> form (i.e. $d\omega|_X = 0$)

And the sym POT is given by:

$$
\begin{tikzcd}
E \arrow[d] & = & {[} T_M|_X \arrow[r, "\mathrm{dow}^\vee"] \arrow[d, "w^\vee"] & \Omega_M|_X {]} \arrow[d, "\mathrm{id}"] \\
\mathbb{L}_X^{\geq -1} & = & {[} I/I^2 \arrow[r, "d"] & \Omega_M|_X {]}
\end{tikzcd}
$$

And $\mathrm{dow}^\vee$ is self dual which gives

$$\theta : E \xrightarrow{\sim} E^\vee[1] \quad ; \quad \theta = \theta^\vee[1]$$

Rmk : Almost closeness is needed for the symmetry : $w = \sum f_i dx_i$ almost closed

$$\Leftrightarrow \quad \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \mod (f_1 \cdots f_n)$$

$$d\omega^\vee : T_M|_x \longrightarrow \Omega_M|_x$$

$$\frac{\partial}{\partial x_i} \longmapsto df_i$$

And a local computation shows :

$$
(d\omega^\vee)^\vee : \quad
\begin{tikzcd}
\Omega_M|_X^\vee \arrow[r] & T_M|_X^\vee \arrow[d, "\wr"] \\
T_M|_X \arrow[u, "\wr"] & \Omega_M|_X
\end{tikzcd}
$$

$$\frac{\partial}{\partial x_i} \longmapsto \sum_j \frac{\partial f_j}{\partial x_i} dx_j$$

$\theta : E \to E^\vee[1]$

$\theta \cong \theta^\vee[1]$

- For every $p \in X$, it's possible to chouse local model for $p \in U \subset X$, $U \hookrightarrow M$ s.t. $\dim M = \dim T_X|_p$.

And recall the construction of virtual class, we need to construct a cone $C \subseteq \Omega_{M}/X$,

and $[X]^{vir} := 0^{!}[C]$

we'll call C the obstruction cone, and in the local picture it's $C_{X/M} = C \hookrightarrow \Omega_{M}/X$

We summarize the proof for $\int_{[X]^{vir}} 1 = \tilde{\chi}(X)$

in a diagram:

$$
\begin{tikzcd}
 & Z_*(M) \arrow[dl, "c_0^M"'] \arrow[dd, "\mathrm{Eu}"] \arrow[dr, shift left, "L"] & \mathcal{C} \arrow[d, phantom, "\cap"] \\
A_0(M) \arrow[d, "\int_M"'] & & \mathcal{L}(\Omega_M) \arrow[ul, shift left, "\pi'"] \arrow[ll, dashed, red, "0^!"'] \\
\mathbb{Q} & \mathrm{Con}(M) \arrow[l, "\chi"] \arrow[ur, "\sim", "ch"'] &
\end{tikzcd}
$$

[sheaves on mfld]

- Solid arrows commute are results from microlocal geometry.

- The red arrow commutes with others is proved in Behrend's paper
- The arrow c^M is irrelavant with our dicussion, but it exists!
- We briefly explain $\mathcal{L}(\Omega_M)$ and the maps

$$Z_*(M) \underset{\pi'}{\overset{L}{\rightleftarrows}} \mathcal{L}(\Omega_M)$$

- Ω_M has tautological 1-form $\lambda = \sum_i p_i \, dx_i$

where $(x_1 \cdots x_n)$ is étale coordinate on M

$(p_1 \cdots p_n)$ are induced vertical coordinate on Ω_M.

- $\mathcal{L}(\Omega_M) \subseteq Z_*(\Omega_M)$ is generated by <u>conic</u> <u>lagrangian</u> cycles, i.e. $V \subseteq \Omega_M$ s.t. <u>$\dim V = n$</u>

<u>$\lambda|_V = 0$</u>

- The maps are defined by

$$L: \underset{\in Z_*(M)}{V} \longmapsto (-1)^{\dim V} [\text{Conormal } V/M]$$

$$\pi': \underset{\text{prime cycle in } \Omega M}{W} \longmapsto (-1)^{\dim \pi(W)} \pi(W)$$

They're inverse to each other.

- $\int_{[X]^{vir}} 1 = \tilde{\chi}(X)$ will follow from the following Theorem:

<u>Thm</u> : The obstruction cone $C_{X/M}$ is conic lagrangian, i.e. $C_{X/M} \in \mathcal{L}(\Omega_M)$.

$\underline{X} \hookrightarrow M$

$Z(\underline{df}) = X$

$$\nu_X(p) = \left(1 - \chi(F_p)\right)(-1)^{\dim M}$$

Part II : MNOP for $\beta = 0$

- For Y proj smooth CY 3fold, $I_n(Y,0) \cong \mathrm{Hilb}^n(Y)$ regarded as moduli of sheaves.
- A Sym POT has been constructed.
- Conj in MNOP may be reformulated as

$$\sum \tilde{\chi}(\mathrm{Hilb}^n(Y))\, t^n = M(-t)^{\chi(Y)}$$

$$M(t) = \prod_i (1-t^i)^{-i}$$

The main technique will be $\mathbb{G}_m$ - localization. The following Prop computes $\nu_X(p)$ for p isolated $\mathbb{G}_m$- fix pt:

Prop: M be a smooth $\mathbb{G}_m$ - scheme, ω an inv 1 - form on M, $p \in M$ be an isolated fixed pt

Suppose further $p \in X = Z(\omega)$, Then

$$\nu_X(p) = (-1)^{\dim M}$$

$\nu_X(p) = (-1)^{\dim M - \dim F} \nu_F(p)$

This can be extend to non isolated case [Zhenbo Qin, ...]

<u>Sketch</u> of the proof : We'll rely on the following <u>facts</u>

- $\nu_x(P) = L_{S_\varepsilon}(\Delta_1 \cap S_\varepsilon , \Gamma_y \cap S_\varepsilon)$ $\quad S_\varepsilon \subseteq \Omega_M$

(ν_x may be computed via "linking number")

- $L_{S_\varepsilon}(A \cap S_\varepsilon , B \cap S_\varepsilon) = (-1)^{\text{orient}} \cdot I_p(A, B)$

if $A \cap B$ only at P in $\overline{B}_\varepsilon \subseteq \Omega_M$.

(Linking number becomes intersection number up to orientation)

- For $S^1 \subseteq \mathbb{G}_m$ Linking number is invariant under S^1-equivariant differentiable homotopy

Now let's define Δ_1 & Γ_η in the first fact,

Choose étale coordinates $(x_1 \cdots x_n)$ around p on M, extend it to induced coordinates $(x_1 \cdots x_n, p_1, \cdots p_n)$ on Ω_M.

Furthermore, assume $\mathbb{G}_m$ acts on $(x_1 \cdots x_n)$ diagonally with weight r_i

- Define $A_t \subseteq \Omega_M$ locally be the section of "slope" t, i.e. by local equations $t p_i = \bar{x}_i$

- Define $\Gamma_\eta \subseteq \Omega_M$ be $\mathrm{Im}\,\frac{1}{\eta}\omega$, or equivalently by local equation $\eta p_i = f_i(x)$

  where $\omega = \Sigma f_i \, dx_i$ is the inv 1-form.

Next consider the homotopy :

$$\mathbb{R} \times S^{2n-1} \longrightarrow S_\varepsilon$$

$$(t, \underset{\underset{-r_1}{\uparrow}}{p_1} \cdots \underset{\underset{-r_n}{\uparrow}}{p_n}) \longmapsto \frac{\varepsilon}{\sqrt{1+t^2}} (t\bar{p}_1, \cdots t\bar{p}_n, p_1 \cdots p_n)$$

its S^1 - equivariant homotopy between

$$\Delta_0 \cap S_\varepsilon \sim \Delta_1 \cap S_\varepsilon$$

So $L_{S_\varepsilon}(\Delta_1 \cap S_\varepsilon, \Gamma_g \cap S_\varepsilon) = L_{S_\varepsilon}(\Delta_0 \cap S_\varepsilon, \Gamma_g \cap S_\varepsilon)$

But Δ_0 is vertical, it's the fibre of Ω_n over p!, so $\Delta_0 \cap \Gamma_\eta = p$ and the intersection is transverse!

p is isolated fixed pt!

So $v_X(p) = L_{S_\varepsilon}(\Delta_0 \cap S_\varepsilon, \Gamma_\eta \cap S_\varepsilon)$

$= (-1)^{\text{orient}} \cdot 1$

$= (-1)^n$ (Here, I won't work out detail about the orientation)

Well, Now you may feel we have the following theorem :

$\boxed{\text{Wrong Statement}}$: Let X be a $\mathbb{G}_m$-scheme with $\mathbb{G}_m$-equivariant POT, and $p \in X$ is isolated fixed pt, Then $\nu_X(p) = (-1)^{\dim T_X|_p}$

comes from we may chose local embedding to have minimal dimension.

The problems are:

- We can't present $X = Z(\omega) \hookrightarrow M$ globally for some $\mathbb{G}_m$-scheme M and inv 1-form ω.

- We might lose the minimal dimension of M in case we insist on $\mathbb{G}_m$-equivariant embedding.

However, the problems can be resolved in <u>affine</u> case, we may prove:

- X affine $\mathbb{G}_m$-scheme, $p \in X$ isolated fixed pt

  Then there is an inv open affine $p \in X' \subseteq X$

  and $\iota : X' \hookrightarrow M$ equivariant into sm $\mathbb{G}_m$-scheme.

  s.t. $\dim M = \dim T_X|_p$ and so $p \in M$

  remains isolated fixed pt.

- For X affine $\mathbb{G}_m$- Scheme with an equivariant Sym POT, for every isolated fixed pt $p \in X$

There is inv affine nbhd $p \in X' \subseteq X$ and

$\iota : X' \hookrightarrow M$ equivariantly into sm M with $\dim T_X|_p$

And ω inv 1-form on M s.t.

$X' = Z(\omega)$ and the POT induced by ω is

the "Same" with the original one.

We omit the proofs. one is commutative algebra, and another is homological algebra.

The important thing is: after impose the affine condition, the wrong statement becomes true:

<u>Thm</u>: X <u>affine</u> $\mathbb{G}_m$-scheme with equivariant Sym POT

$p \in X$ is isolated fixed pt, Then:

$$\nu_X(p) = (-1)^{\dim T_X|_p}$$

Combined with $\chi(\mathbb{G}_m) = 0$, we get the useful corollary:

$v_x(\text{orbit}) = \text{const}$

<u>Cor</u>: For X $\mathbb{G}_m$-scheme, with all fixed pts isolated, and around every of them there is an inv affine open which over there exist Sym POT, Then:

$$\tilde{\chi}(X) = \sum_p (-1)^{\dim T_X|_p}$$

(sum over all fixed pts on X).

Moreover, For $Z \subseteq X$ inv locally subset

$$\tilde{\chi}(Z, X) = \sum_{p \in Z} (-1)^{\dim T_X|_p}$$

(sum over all fixed pts in Z)

We'll apply the method to Hilb$^n(\mathbb{A}^3)$ next:

- Consider $T = \mathbb{G}_m^3 \curvearrowright \mathbb{A}^3$ diagonally with weight 1
- Take $T_0 \subseteq T$ be two dimensional torus

$$\{ (t_1, t_2, t_3) \mid t_1 t_2 t_3 = 1 \}$$

For $T_0 \curvearrowright \mathbb{A}^3$, its induced action $T_0 \curvearrowright \mathrm{Hilb}^n \mathbb{A}^3$, we have:

Lemma: (a): There are finitely many fixed pts. They correspond to monomial ideals.

(b) For I such an ideal, Let $d = \dim T_I \mathrm{Hilb}^n \mathbb{A}^3$

Then $(-1)^d = (-1)^n$

Pf: (a) is easy, (b) is by localization, mentioned in previous talks.

Prop: For any T_0-inv locally closed subset $Z \subseteq \mathrm{Hilb}^n \mathbb{A}^3$: $\tilde{\chi}(Z, \mathrm{Hilb}^n \mathbb{A}^3) = (-1)^n \chi(Z)$

Pf: This follows from two technical results:

- There is an one para subgp $\mathbb{G}_m \subseteq T_0$

s.t. $\mathbb{G}_m \curvearrowright \mathrm{Hilb}^n \mathbb{A}^3$ won't produce more fixed pts. ??

- For above $\mathbb{G}_m$ action, we can find inv affine open around each fixed pt where sym POT exists.

(This follows from we have sym POT on $\mathrm{Hilb}^n(\mathbb{A}^3) \hookrightarrow \mathrm{Hilb}^n(\mathbb{P}^3)$ induced by T_0-inv anticanonical section, and also a $\mathbb{G}_m$-inv ample line bd)

Cor : Let $F_n \subseteq \mathrm{Hilb}^n \mathbb{A}^3$ be closed subset consists of subschemes supported at the origin.

clearly F_n is T_0-inv, we have :

$$\tilde{\chi}(F_n, \mathrm{Hilb}^n \mathbb{A}^3) = (-1)^n \chi(F_n) = (-1)^n \# \text{ fixed pts}$$

$= (-1)^n \cdot \#$ monomial ideal of length n

So $$\sum_{n=0}^{\infty} \tilde{\chi}(F_n, \mathrm{Hilb}^n \mathbb{A}^3) t^n = M(-t)$$

Finally, we introduce the stratification argument to conclude:

Let Y be a smooth 3 fold (Not necessarily proj)

For any partition $\alpha = (\alpha_1 \cdots \alpha_r)$ of n, we have

$\mathrm{Hilb}^n_\alpha Y \subseteq \mathrm{Hilb}^n Y$ be a strata of subschemes

supported at n pts with length α_i s.

$$\begin{tikzcd}
Z_2 \arrow[r] \arrow[d, "\text{Galois}"'] \arrow[dr, phantom, "\square"] & V \arrow[r] \arrow[d, "f_2"] & \prod \mathrm{Hilb}^{d_i} Y \\
\mathrm{Hilb}^n_2 Y \arrow[r] & U \arrow[r] & \mathrm{Hilb}^n Y
\end{tikzcd}$$

- Here V parametrizes r-tuple of subschemes with pairwise disjoint supports

- Let $f_2 : V \longrightarrow \mathrm{Hilb}^n Y$ be the obvious map, and

  f_2 is étale

- Let $U := \mathrm{Im}(f_2)$, and clearly $\mathrm{Hilb}^n_2 Y \subseteq U$

- Denote $Z_\alpha = f_\alpha^{-1}(\mathrm{Hilb}^n_\alpha Y)$ we have:

$$\begin{tikzcd} Z_\alpha \arrow[r] \arrow[d, "\text{Galois}"'] \arrow[dr, phantom, "\square"] & V \arrow[r] \arrow[d] & \prod \mathrm{Hilb}^{\alpha_i} Y \\ \mathrm{Hilb}^n_\alpha Y \arrow[r] & U \arrow[r] & \mathrm{Hilb}^n Y \end{tikzcd}$$

The morphism $Z_\alpha \longrightarrow \mathrm{Hilb}^n_\alpha Y$ is Galois

with Galois gp $G_\alpha \cong \mathrm{Aut}(\alpha)$

- Further more $Z_\alpha \subseteq \prod \mathrm{Hilb}^{\alpha_i}_{(d_i)} Y$ and we have cartisian diagram :

$$\begin{tikzcd}
Z_\alpha \arrow[r] \arrow[d] & \prod \mathrm{Hilb}^{\alpha_i}_{(d_i)} Y \arrow[d] \\
Y^r_0 \arrow[r] & Y^r
\end{tikzcd}$$

- $Y^r_0 \subseteq Y$ is open subscheme with pairwise disjoint entries.

- $Z_\alpha \to Y^r_0$ is Zariski - locally trivial fibration with fibre $\prod F_{\alpha_i}$

And the Behrend's function for this fibration can be characterized explicitly:

Prop: $\pi_Y : \underline{\mathrm{Hilb}^n_{(n)} Y} \longrightarrow Y$ the morphism induced by Hilbert-Chow, It's Zariski-fibration with fibre $\underline{F_n}$, and there exists Zariski open cover $\{U_i\} \to Y$ s.t.

$$\left(\pi_Y^{-1}(U_i), \nu_Y\right) = (U_i, 1) \times (F_n, \nu_n)$$

Now we may prove the theorem :

<u>Thm</u> : For Y smooth 3 fold , for all $n > 0$

$$\tilde{\chi}(\mathrm{Hilb}^n Y) = (-1)^n \chi(\mathrm{Hilb}^n Y)$$

Thus $$\sum_{n=0}^{\infty} \tilde{\chi}(\mathrm{Hilb}^n Y)\, t^n = M(-t)^{\chi(Y)}$$

$$
\begin{tikzcd}
Z_\lambda \arrow[r] \arrow[d, "\text{Galois}"'] \arrow[dr, phantom, "\square"] & V \arrow[r] \arrow[d] & \prod \mathrm{Hilb}^{\lambda_i} Y \\
\mathrm{Hilb}^n_\lambda Y \arrow[r] & U \arrow[r] & \mathrm{Hilb}^n Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
Z_\lambda \arrow[r] \arrow[d] & \prod \mathrm{Hilb}^{\lambda_i}_{(\lambda_i)} \arrow[d] \\
Y^r_0 \arrow[r] & Y^r
\end{tikzcd}
$$

pf :
$$\tilde{\chi}(\mathrm{Hilb}^n Y) = \sum_{\lambda \vdash n} \tilde{\chi}(\mathrm{Hilb}^n_\lambda Y, \mathrm{Hilb}^n Y)$$

$$= \sum_{\lambda \vdash n} \tilde{\chi}(\mathrm{Hilb}^n_\lambda Y, U) = \sum_{\lambda \vdash n} |G_\lambda| \, \tilde{\chi}(Z_\lambda, V)$$

$$= \sum_{\lambda \vdash n} |G_\lambda| \, \tilde{\chi}(Z_\lambda, \prod \mathrm{Hilb}^{\lambda_i} Y)$$

$$= \sum_{\alpha \vdash n} |G_\alpha| \, \chi(Y_0^{\ell(\alpha)}) \prod_i \tilde{\chi}(F_{\alpha_i})$$

$$= \sum_{\alpha \vdash n} |G_\alpha| \, \chi(Y_0^{\ell(\alpha)}) \prod_i (-1)^{\alpha_i} \chi(F_{\alpha_i})$$

$$= (-1)^n \cdot \sum_{\alpha \vdash n} |G_\alpha| \, \chi(Y_0^{\ell(\alpha)}) \prod_i \chi(F_{\alpha_i})$$

$$= (-1)^n \chi(\mathrm{Hilb}^n Y, \mathbb{1})$$

$$= (-1)^n \chi(\mathrm{Hilb}^n Y) .$$

[C]